

Filtered objects

Goal: Deform filtered objects to their associated graded objects

$k =$ (discrete) comm. ground ring

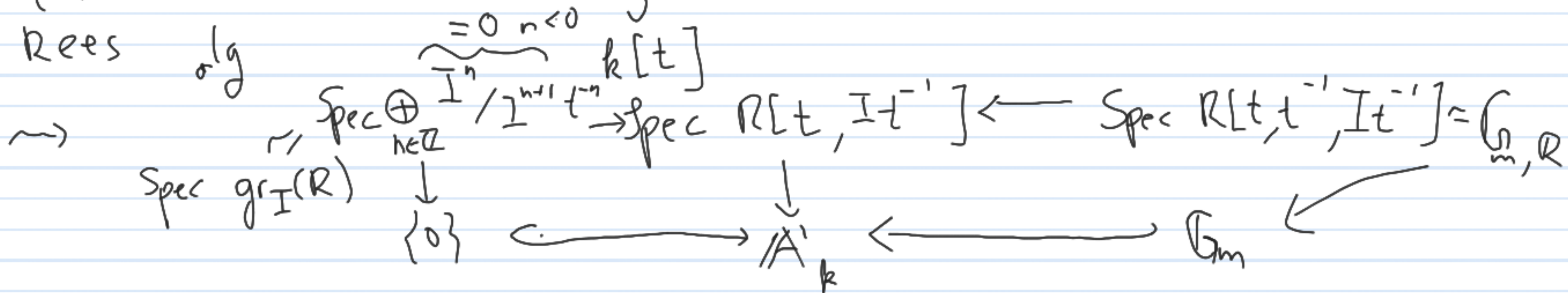
Classical idea (Rees, Deligne, Simpson)

$R =$ comm ring, $I \subset R$ (\rightsquigarrow filtr. $I^n \subset I^{n-1} \subset \dots$)

$\rightsquigarrow R[t, It^{-1}] := \bigoplus_{n \in \mathbb{Z}} I^n \cdot t^{-n} \subset R[t, t^{-1}]$ ($I^n = R$ for $n \leq 0$)

(extended)

Rees alg



Leads to local-graded analogy for many results in comm alg.

Example (Local) Nakayama

R local ring, M fin. gen R -mod.

Then $mM = M \Rightarrow M = 0$

Graded Nakayama

R $\mathbb{Z}_{\geq 0}$ -graded ring, $R_+ \subset R$ irrelevant ideal

M fin. gen. \mathbb{Z} -graded R -mod

$R_+ M = M \Rightarrow M = 0$

Exer Using (slight modification) Rees algebra degeneration from above to deduce local from graded Nakayama

§1 Graded and filtered objects in $\mathcal{D}(k)$ (use ∞ -cat enhancements throughout)

Recall $\mathcal{DF}(k) := \text{Fun}((\mathbb{Z}, \leq)^{\text{op}}, \mathcal{D}(k))$ filtered derived cat

objects: $(\dots \rightarrow K_n \rightarrow K_{n-1} \rightarrow \dots)$ (+ coherence data)

$\mathcal{D}(k)^{\mathbb{Z}\text{-gr}} := \text{Fun}(\mathbb{Z}^{\text{disc}}, \mathcal{D}(k))$ \mathbb{Z} -graded derived cat

objects: $(K_n)_{n \in \mathbb{Z}}$

$\text{colim} : \mathcal{DF}(k) \rightarrow \mathcal{D}(k)$ underlying object
 $(\dots \rightarrow K_n \rightarrow K_{n-1} \rightarrow \dots) \mapsto \text{colim } K_n$

Day convolution defines symm. monoidal structures (check!)

$\partial_n \mathcal{D}(k)^{\mathbb{Z}\text{-gr}}$ $(K \otimes L)_n = \bigoplus_{l+m=n} (K_l \otimes L_m)$ unit k $k_n = \begin{cases} k & n=0 \\ 0 & n \neq 0 \end{cases}$

$\mathcal{DF}(k)$ $(K \otimes L)_n = \text{colim}_{n \leq l+m} (K_l \otimes L_m)$ unit k $k_n = \begin{cases} k & n \leq 0 \\ 0 & n > 0 \end{cases}$

Two functors $\mathcal{DF}(k) \rightarrow \mathcal{D}(k)^{\mathbb{Z}\text{-gr}}$:

A) Inclusion $\mathbb{Z}^{\text{disc}} \hookrightarrow (\mathbb{Z}, \leq)$ induces lax symm. mon. functor

res:
$$\begin{array}{ccc}
 \mathcal{D}\mathcal{F}(k) & \longrightarrow & \mathcal{D}(k)^{\mathbb{Z}\text{-gr}} \\
 \downarrow \scriptstyle{2} & \searrow \scriptstyle{(K_n \rightarrow K_{n-1} \rightarrow \dots)} \longmapsto & \int (K_n \cdot t^{-n}) \\
 & & \text{Mod}_{k[t]}(\mathcal{D}(k)^{\mathbb{Z}\text{-gr}})
 \end{array}$$

$$k \mapsto k[t] \quad k[t]_n = \begin{cases} k & n \leq 0 \\ 0 & n > 0 \end{cases}$$

B) Associated graded

$$\begin{array}{ccc}
 & \text{Mod}_{k[t]}(\mathcal{D}(k)^{\mathbb{Z}\text{-gr}}) & \xrightarrow{\oplus} k \\
 \nearrow \text{res} & & \downarrow \scriptstyle{k[t]} \\
 \mathcal{D}\mathcal{F}(k) & \longrightarrow & \mathcal{D}(k)^{\mathbb{Z}\text{-gr}} \\
 (\dots \rightarrow K_n \rightarrow K_{n-1} \rightarrow \dots) & \longmapsto & (\text{cofib}(K_{n+1} \rightarrow K_n))_{n \in \mathbb{Z}}
 \end{array}$$

Goal: geometric interpretation

Ex $D(k)^{\mathbb{Z}\text{-gr}} \simeq \text{QCoh}(B\mathbb{G}_m)$ "b/c \mathbb{G}_m diagonalizable"
 "eigen sheaf decomp" w/ $\widehat{\mathbb{G}}_m = \mathbb{Z}$

(Warning $\text{QCoh}(X) = \lim_{\text{Spec } A \rightarrow X} D(R)$ DAG notation for $D_{\text{qcoh}}(X)$)

"Affine deformation" of $B\mathbb{G}_m$

$$\{0\} \longleftarrow A' \longleftarrow \mathbb{G}_m \rightsquigarrow B\mathbb{G}_m \xrightarrow{0} [A'/\mathbb{G}_m] \longleftarrow [\mathbb{G}_m/\mathbb{G}_m] \simeq *$$

$$\downarrow \mathbb{G}_m$$

Thm (Simpson, Moulinos). \exists comm. diagram

$$\begin{array}{ccccc} \text{QCoh}(B\mathbb{G}_m) & \xleftarrow{0^*} & \text{QCoh}(A'/\mathbb{G}_m) & \xrightarrow{1^*} & \text{QCoh}(*) \\ \downarrow S & & \downarrow S & & \parallel \\ D(k)^{\mathbb{Z}\text{-gr}} & \xleftarrow{\text{gr}} & D^{\text{fl}}(k) & \xrightarrow{\text{colim}} & D(k) \end{array}$$

in which the vertical functors are symm. mon. equivalences

Pf idea: $\cdot)$ $[A'/G_m] \rightarrow BG_m$ affine $\Rightarrow \text{QCoh}(A'/G_m) \simeq \text{Mod}_{\pi_* \mathcal{O}_{A'/G_m}}(\text{QCoh}(BG_m))$

$$\left(\pi_* \mathcal{O}_{A'/G_m} \simeq k[t] \right)$$

by flat base change along

$$\begin{array}{ccc} A' & \rightarrow & A'/G_m \\ \downarrow & & \downarrow \\ * & \rightarrow & BG_m \end{array}$$

and comparison of $k[t^{\pm 1}]$ -module structures

$$\text{Mod}_{k[t]}(\mathbb{D}(k)^{\mathbb{Z}\text{-gr}}) \xrightarrow{\cong} \text{DF}(k)$$

$\cdot)$ \mathcal{O}^* given by $\otimes_{k[t]} k \rightsquigarrow$ left square commutes

$$\cdot)$$
 $(K_n)_{n \in \mathbb{Z}} \in \text{Mod}_{k[t]}(\mathbb{D}(k)^{\mathbb{Z}\text{-gr}})$

$\text{fib}(K_n \rightarrow K_n[\frac{1}{t}])$ is t^∞ -torsion

$$\rightsquigarrow \text{colim}(K_n \xrightarrow{t} K_{n-1} \rightarrow \dots) = \text{colim}(K_n[\frac{1}{t}] \xrightarrow{t} K_{n-1}[\frac{1}{t}] \rightarrow \dots)$$

\rightsquigarrow right square commutes

Rem \exists version / Spec(B) (Moulinos)

Def graded stack = stack / B \mathbb{G}_m
filtered stack = stack / [A'/ \mathbb{G}_m]

Def Let $X \rightarrow [A'/\mathbb{G}_m]$ filtered stack. Then get diagram
ass. graded stack \curvearrowright $X^{gr} \rightarrow X \leftarrow X^u$ underlying stack
 $\downarrow \pi^{gr}$ $\downarrow \pi$ $\downarrow \pi^u$ $(*)$
B $\mathbb{G}_m \rightarrow [A'/\mathbb{G}_m] \leftarrow *$

Def $X \xrightarrow{\pi} [A'/\mathbb{G}_m] \rightsquigarrow \mathcal{O}^{fil}(X) := \pi_* \mathcal{O}_X \in DF(k)$

If have base change for squares in $(*)$
associated graded of $\mathcal{O}^{fil}(X) = (\pi^{gr})_* \mathcal{O}_{X^{gr}}$
underlying object $= \pi^u_* \mathcal{O}_{X^u}$

Base change ensured by X qcqs

(A) π of fin. coh. dim

$$\forall R \in \text{CALg}_k^{an}, \text{Spec } R \rightarrow [A'/G_m]$$

$$\pi_{R,*} \text{QCoh}(X \times_{A'/G_m} \text{Spec } R)_{\geq 0} \rightarrow D(R)_{\geq n} \text{ for some } n$$

(B) X admits flat hypercover by affines w/ flat transition maps

\rightsquigarrow base change $\begin{matrix} \text{atlas} \\ \text{for right } \square \\ \text{left} \end{matrix}$ $\text{QCoh}(X)_{\infty}$
 $\text{QCoh}(X)$

Classical situation: X normal variety, $X \rightarrow [A'/G_m]$ flat

$$\Leftrightarrow L \in \text{Pic}(X) + \text{set } H^0(X, L) \text{ for } n \geq 1$$

$\rightarrow D \subset X$ Cartier div.

Filtration $\mathcal{O}^{\text{fil}}(X)$

$$= \text{Spec}_X(\bigoplus L^{-n})$$

$$\begin{array}{ccc} V(L) \cap 0 & \rightarrow & X \\ \downarrow & & \downarrow \\ X & & X \end{array}$$

$$A' \xrightarrow{\mathcal{G}} [A'/G_m]$$

$$D \rightarrow X \longleftarrow X \setminus D$$

$$BG_m \rightarrow [A'/G_m] \longleftarrow *$$

\rightsquigarrow filtration on $\Gamma(X \setminus D, \mathcal{O}_{X \setminus D})$

can be described, $D \subset X$ Cartier div \rightsquigarrow concretely

$$\rightsquigarrow \mathcal{G}^{-1} \mathcal{O}^{\text{fil}}(X) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, L^{-n}) \text{ (derived)}$$

associated graded = $\bigoplus_{n \in \mathbb{Z}} \Gamma(D, L|_D^{-n})$

underlying object = $\Gamma(X \setminus D, L|_{X \setminus D})$

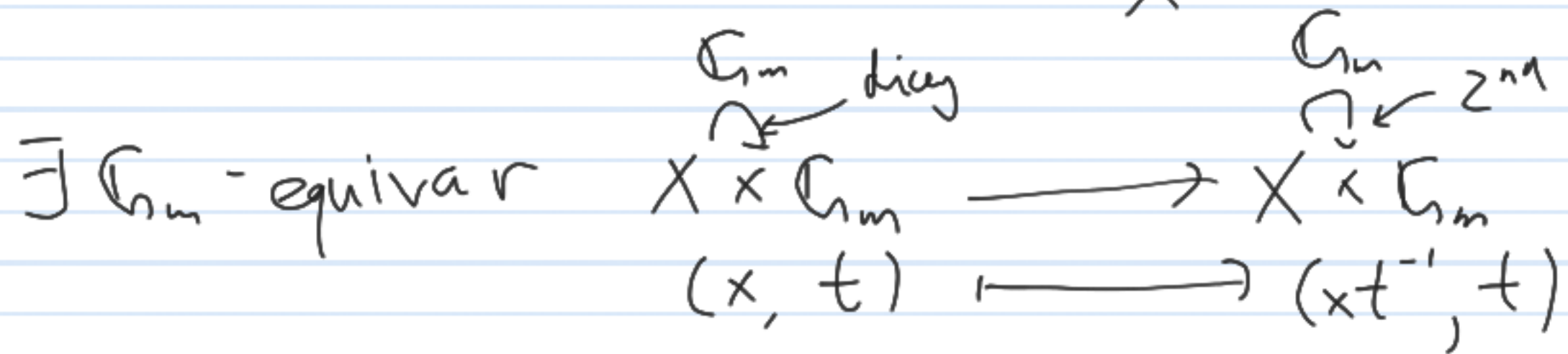
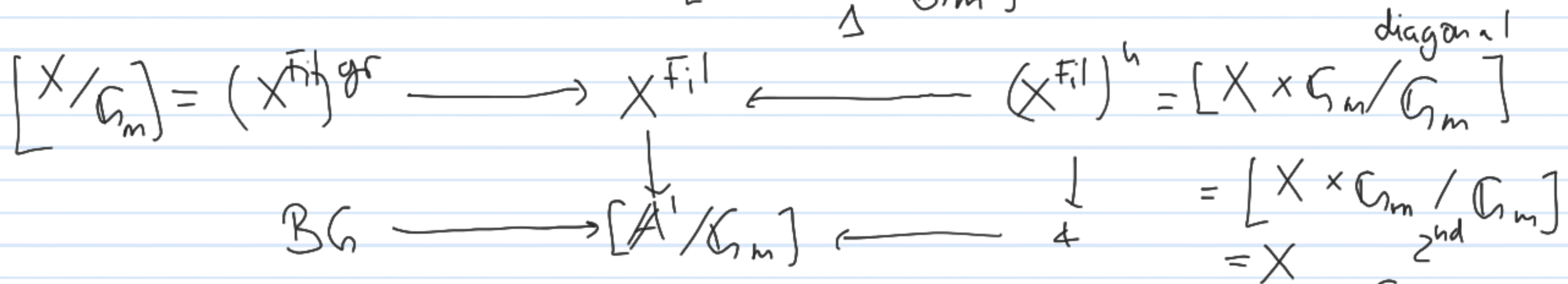
$$\xrightarrow{\mathcal{G}^{-1}} \Gamma(X \setminus D, \mathcal{O})$$

$$\mathcal{O}_X(*D) \xrightarrow{\sim} \mathcal{O}_{X \setminus D} \text{ has pole-order filter.}$$

$$\bigcup \mathcal{O}_X(nD)$$

Ex ("split filtered stack") X stack, $G_m \curvearrowright X$

\rightsquigarrow filtered stack $X^{Fil} := [X \times A^1 / G_m]$



II Applications to Witt vectors

Fix prime p , $k = \mathbb{Z}_{(p)}$ Witt vectors = p -typical Witt vectors

Frob: $W \rightarrow W \rightsquigarrow$ subgroup schemes Ker, Fix

Goal: Construct filtration on Fix w/ associated graded $[\text{Ker}/G_m]$

§1 Review Witt vectors

Def (Joyal) A S -ring is a pair (R, S) where

- R comm. ring
- $S: R \rightarrow R$ map subject to some relations ensuring

Frob: $x \mapsto x^p + p \cdot S(x)$ is a ring homom.

Fact: \exists adjoint functors $(S\text{-Rings}) \begin{matrix} \xleftarrow{\text{Free}} \\ \xrightarrow{\text{Forget}} \\ \xleftarrow{W} \end{matrix} (\text{Rings})$

$$1) \text{ Free } (k[x]) = k\{x\} = k[x_0, x_1, x_2, \dots]$$

$$\sigma(x_i) = x_{i+1}$$

2) As sets,

$$W(\mathbb{R}) = \text{Hom}_{\text{Ring}}(k[x], W(\mathbb{R})) = \text{Hom}_{\mathcal{S}\text{-ring}}(k\{x\}, W(\mathbb{R}))$$

$$\downarrow$$

$$W_n(\mathbb{R}) \longrightarrow \text{Hom}_{\text{Ring}}(k[x_0, \dots, x_n], \mathbb{R}) = \mathbb{R}^{n+1}$$

$$= \text{Hom}_{\text{Ring}}(k\{x\}, \mathbb{R}) = \mathbb{R}^{\mathbb{N}}$$

3) Have ring morphism

$$\text{Ghost} : W(\mathbb{R}) \longrightarrow \mathbb{R}^{\mathbb{N}}$$

$$x \longmapsto (x, \phi(x), \phi^2(x), \dots)$$

$$x^p \longmapsto x^p + p\sigma(x)$$

Thm (Witt) Existence of Ghost uniquely determines ϕ on $W(R)$
 (functorially in R)

$$4) \quad \forall \mathbb{Q} : \delta(x) = \frac{\phi(x) - x^p}{p}$$

$$\rightsquigarrow \forall \mathbb{Q}\text{-alg } R, \text{ Ghost} : \begin{array}{ccc} W(R) & \xrightarrow{\sim} & R^{\mathbb{N}} \\ \downarrow & & \downarrow \\ W_n(R) & \longrightarrow & R^{n+1} \end{array}$$

5) \exists multiplicative Teichmüller map $R \longrightarrow W(R) \stackrel{\text{sets}}{\cong} R^{\mathbb{N}}$

$$\text{Everything functorial in } R \quad x \longmapsto (x, 0, 0, \dots) = [x]$$

\rightsquigarrow comm. group scheme $W \cong \varinjlim W_n \stackrel{\text{as schemes}}{\cong} A^{n+1}$
 + Frobenius $\text{Frob} : W \longrightarrow W$
 $\downarrow \quad \downarrow$
 $W_{n+1} \longrightarrow W_n$
 + Teichmüller map $[] : G_m \longrightarrow W$
 + Ghost maps $W \longrightarrow \prod G_a$

§2 Ker and Fix

Def $\text{Fix} = \ker \left(W \xrightarrow{\text{Frob} - \text{id}} W \right)$

$\text{Ker} = \ker \left(W \xrightarrow{\text{Frob}} W \right)$

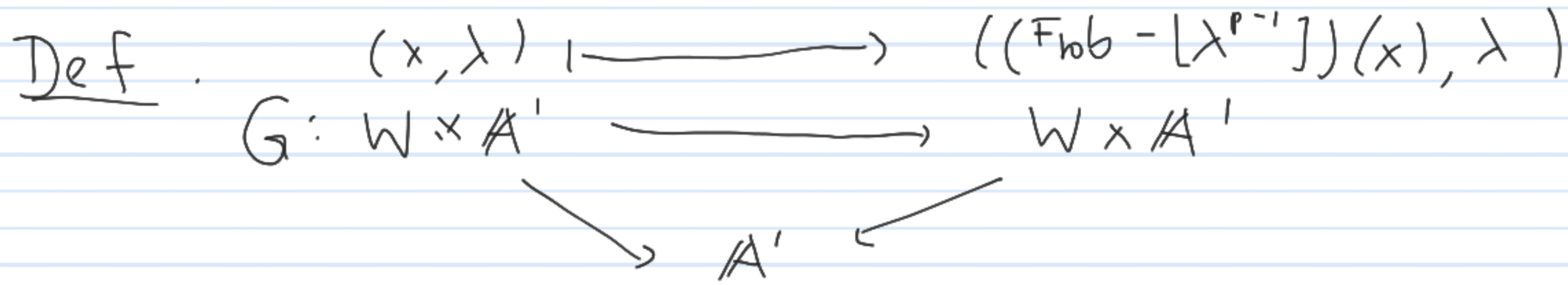
/ \mathbb{Q} : Ghost:

$$\begin{array}{ccc}
 W_{\mathbb{Q}} & \xrightarrow{\sim} & \prod \mathbb{C}_{r, \mathbb{Q}} (z_0, z_1, z_2, \dots) \\
 \text{Frob} - \text{id} \downarrow & & \downarrow \\
 W_{\mathbb{Q}} & \xrightarrow{\sim} & \prod \mathbb{C}_{r, \mathbb{Q}} (z_1 - z_0, z_2 - z_1, z_3 - z_2, \dots)
 \end{array}$$

$\rightsquigarrow \text{Fix}_{\mathbb{Q}} \cong \mathbb{C}_{r, \mathbb{Q}} \xrightarrow{\Delta} \prod \mathbb{C}_{r, \mathbb{Q}}$

Likewise, $\text{Ker}_{\mathbb{Q}} \cong \mathbb{C}_{r, \mathbb{Q}} \xrightarrow{\text{incl}_0} \prod \mathbb{C}_{r, \mathbb{Q}}$

Goal: Interpolate b/w Fix and Ker away from \mathbb{Q}



$$\leadsto (\ker G) \subset W \times A'$$

$$\cdot (\ker G)_\lambda = \text{Ker}$$

Check: $\ker G$ is G_m -stable for diagonal action (via $[]$ on first comp.)

$$\therefore \text{For } \lambda \in A^\times, (\ker G)_\lambda \cong (\ker G)_1 \cong \ker(\text{Frob} - [1]) \cong \text{Fix}$$

$$(x, \lambda) \mapsto \left(\left[\frac{1}{\lambda} \right] x, 1 \right)$$

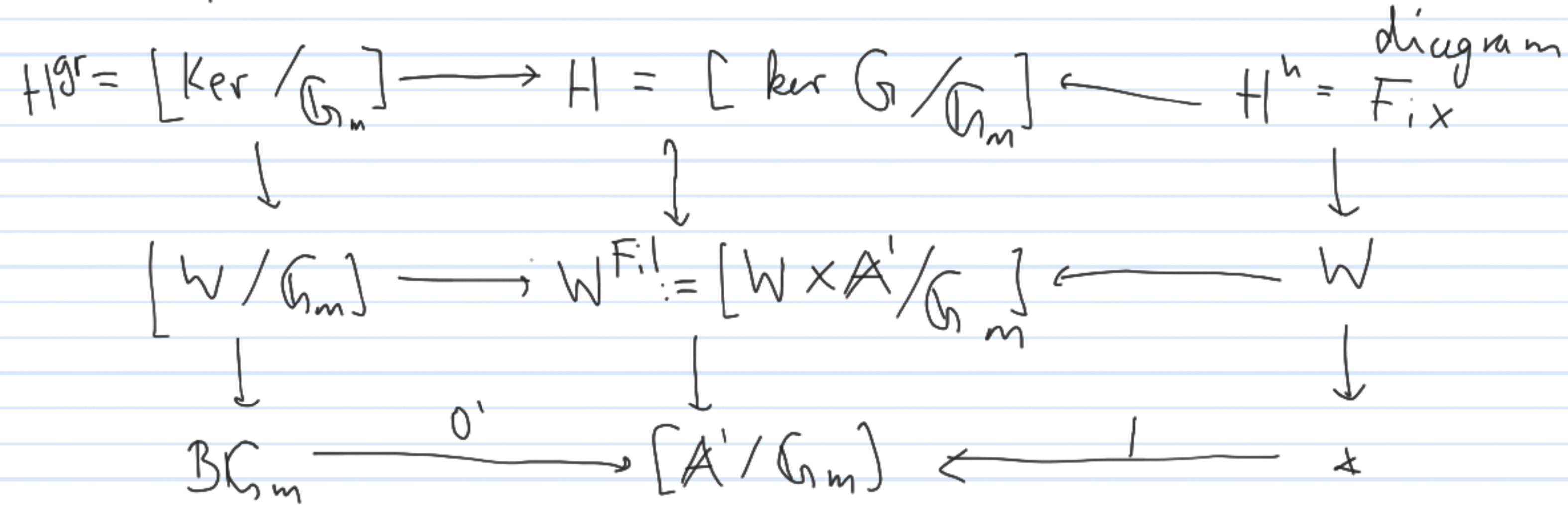
$$\underbrace{G}_{([]_{\text{nat}}^{-1})}: W \times A' \rightarrow$$

$$\underbrace{W \times A'}_{([]^p, \text{nat}^{-1})}$$

$$(x, \lambda) \mapsto (\text{Frob}(x) - [\lambda^{p-1}]x, \lambda) = [\lambda^p] \text{Frob}(x) - [\lambda^p] [\lambda^{p-1}]x, \lambda$$

Apply split filtered stack construction to obtain fiber product

Def



Lemma $G: W \times A' \longrightarrow W \times A'$ is an fpqc cover, in part,

have SES of group schemes $0 \longrightarrow ker G \longrightarrow W \times A' \xrightarrow{G} W \times A' \longrightarrow 0$.

Cor $H \longrightarrow [A' / G_m]$ flat

Pf $ker G \longrightarrow W \times A' \xrightarrow{G} W \times A' \implies ker G \longrightarrow A'$ flat $\implies [ker G / G_m] \xrightarrow{=} H \longrightarrow [A' / G_m]$ flat \square

Idea of pt of Lemma

In \exists factorization

$$\begin{array}{ccc} W \times A' & \xrightarrow{G} & W \times A' \\ \downarrow & & \downarrow \\ W_{n+1} \times A' & \longrightarrow & W_n \times A' \end{array}$$

Limit argument

STS $W_{n+1} \times A' \longrightarrow W_n \times A'$ is fpqc cover

Critère de platitude par fibres

\Rightarrow STS $(W_{n+1} \times A')_{\lambda} \longrightarrow (W_n \times A')_{\lambda}$ fpqc cover
 $\forall \lambda \in A' = A'_{\mathbb{Z}(p)}$

Compatibility w/ G_n -action \Rightarrow Enough to check $\lambda = (T), (p, T), (T-1), (p, T-1)$

i.o., $W_{n+1, \mathbb{Q}} \xrightarrow{\text{Frob}} W_n, \mathbb{Q}$ $W_{n+1, \mathbb{F}_p} \xrightarrow{\text{Frob}} W_n, \mathbb{F}_p$ \checkmark Kuntz
 $W_{n+1, \mathbb{Q}} \xrightarrow{\text{Frob-id}} W_n, \mathbb{Q}$ $W_{n+1, \mathbb{F}_p} \xrightarrow{\text{Frob-id}} W_n, \mathbb{F}_p$ \checkmark Artin-Schreier

/ \mathbb{Q} : can identify w/ $\prod G_{n, \mathbb{Q}} \longrightarrow \prod G_{n, \mathbb{Q}}$
 $(z_0, z_1, z_{11}, \dots) \longmapsto (z_0, z_1, z_{11}, \dots) / (z_1 - z_0, z_2 - z_{11}, \dots)$ □