Filtered objects

<u>Goal</u>: Deform filtered objects to their associated graded objects

k = (discrete) comm. ground ring

<u>Classical idea</u> (Rees, Deligne, Simpson)

R = comm ring, $I \subset R$ ($\rightsquigarrow$ filtr. $I^n \subset I^{n-1} \subset \dots$)

$\rightsquigarrow R[t, It^{-1}] := \bigoplus_{n\in\mathbb{Z}} I^n \cdot t^{-n} \subset R[t,t^{-1}]$ ($I^n = R$ for $n \leq 0$)

(extended) Rees alg $\qquad k[t]$

$$\rightsquigarrow \quad \begin{array}{ccccc}
\operatorname{Spec} \bigoplus_{n\in\mathbb{Z}} \underbrace{I^n}_{=0\ n<0}/I^{n+1}t^{-n} & \longrightarrow & \operatorname{Spec} R[t, It^{-1}] & \longleftarrow & \operatorname{Spec} R[t,t^{-1},It^{-1}] \cong \mathbb{G}_{m,R} \\
\downarrow & & \downarrow & & \swarrow \\
\{0\} & \hookrightarrow & \mathbb{A}^1_k & \hookleftarrow & \mathbb{G}_m
\end{array}$$

$\cong \operatorname{Spec} gr_I(R)$

Leads to local-graded analogy for many results in comm alg.

<u>Example</u>

<u>(Local) Nakayama</u>

R local ring, M fin. gen R-mod

Then $\mathfrak{m}M = M \Rightarrow M = 0$

<u>Graded Nakayama</u>

R $\mathbb{Z}_{\geq 0}$-graded ring, $R_+ \subset R$ irrelevant ideal

M fin. gen. $\mathbb{Z}$-graded R-mod

$R_+ M = M \Rightarrow M = 0$

<u>Exer</u> Using (slight modification) Rees algebra degeneration from above to deduce local from graded Nakayama

§1 <u>Graded and filtered objects in $D(k)$</u>

(use ∞-cat enhancements throughout)

<u>Recall</u> $DF(k) = \mathrm{Fun}((\mathbb{Z}, \leq)^{op}, D(k))$ filtered derived cat

objects: $(\dots \to K_n \to K_{n-1} \to \dots)$ (+ coherence data)

$D(k)^{\mathbb{Z}\text{-gr}} := \mathrm{Fun}((\mathbb{Z}^{disc}), D(k))$ $\mathbb{Z}$-graded derived cat

<u>objects</u>: $(K_n)_{n \in \mathbb{Z}}$

$\mathrm{colim} : DF(k) \longrightarrow D(K)$ underlying object

$(\cdots \to K_n \to K_{n-1} \to \cdots) \mapsto \mathrm{colim}\, K_n$

Day convolution defines symm. monoidal structures (check!)

On $D(k)^{\mathbb{Z}\text{-gr}}$ $(K \otimes L)_n = \bigoplus_{l+m=n} (K_l \otimes L_m)$ unit k $k_n = \begin{cases} k & n=0 \\ 0 & n \neq 0 \end{cases}$

$DF(k)$ $(K \otimes L)_n = \mathrm{colim}_{n \leq l+m} (K_l \otimes L_m)$ unit k $k_n = \begin{cases} k & n \leq 0 \\ 0 & n > 0 \end{cases}$

<u>Two functors $DF(k) \to D(k)^{\mathbb{Z}\text{-gr}}$</u>:

A) Inclusion $\mathbb{Z}^{disc} \hookrightarrow (\mathbb{Z}, \leq)$ induces lax symm. mon. functor

$$\begin{tikzcd}
\mathrm{res}: & DF(k) \arrow[r] \arrow[dr, "\wr"'] & D(k)^{\mathbb{Z}\text{-gr}} \\
& & \mathrm{Mod}_{k[t]}\left(D(k)^{\mathbb{Z}\text{-gr}}\right)
\end{tikzcd}$$

$$(K_n \to K_{n-1} \to \dots) \longmapsto \int (K_n \cdot t^{-n})$$

$k \mapsto k[t] \quad k[t]_n = \begin{cases} k & n \le 0 \\ 0 & n > 0 \end{cases}$

B) Associated graded

$$\begin{tikzcd}
& \mathrm{Mod}_{k[t]}\left(D(k)^{\mathbb{Z}\text{-gr}}\right) \arrow[dr, "\otimes_{k[t]} k"] & \\
DF(k) \arrow[ur, "\mathrm{res}"] \arrow[rr] & & D(k)^{\mathbb{Z}\text{-gr}}
\end{tikzcd}$$

$$(\dots \to K_n \to K_{n-1} \to \dots) \longmapsto \left(\mathrm{cofib}\,(K_{n+1} \to K_n)\right)_{n \in \mathbb{Z}}$$

<u>Goal</u>: geometric interpretation

Ex $D(k)^{\mathbb{Z}\text{-gr}} \simeq QCoh(B\mathbb{G}_m)$ b/c $\mathbb{G}_m$ diagonalizable
"eigensheaf decomp" w/ $\hat{\mathbb{G}}_m = \mathbb{Z}$

(Warning $QCoh(X) = \lim_{\mathrm{Spec} A \to X} D(R)$ DAG notation for $D_{qcoh}(X)$)

"Affine deformation" of $B\mathbb{G}_m$

$$\{0\} \hookrightarrow \underset{\overset{\circlearrowleft}{\mathbb{G}_m}}{\mathbb{A}^1} \hookleftarrow \mathbb{G}_m \quad \rightsquigarrow \quad B\mathbb{G}_m \xhookrightarrow{0} [\mathbb{A}^1/\mathbb{G}_m] \xhookleftarrow{1} [\mathbb{G}_m/\mathbb{G}_m] \simeq *$$

Thm (Simpson, Moulinos) $\exists$ comm. diagram

$$\begin{tikzcd}
QCoh(B\mathbb{G}_m) \arrow[d, "\wr"] & QCoh(\mathbb{A}^1/\mathbb{G}_m) \arrow[l, "0^*"'] \arrow[r, "1^*"] \arrow[d, "\wr"] & QCoh(*) \arrow[d, equal] \\
D(k)^{\mathbb{Z}\text{-gr}} & DF(k) \arrow[l, "gr"'] \arrow[r, "colim"] & D(k)
\end{tikzcd}$$

in which the vertical functors are symm. mon. equivalences

Pf idea: ·) $[\mathbb{A}^1/\mathbb{G}_m] \to B\mathbb{G}_m$ affine $\Rightarrow QCoh([\mathbb{A}^1/\mathbb{G}_m]) \simeq \mathrm{Mod}_{\pi_* \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}}(QCoh(B\mathbb{G}_m))$

$\quad \simeq \mathrm{Mod}_{k[t]}(D(k)^{\mathbb{Z}\text{-gr}}) \simeq DF(k)$

($\pi_* \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m} \simeq k[t]$

by flat base change along
$$\begin{array}{ccc} \mathbb{A}^1 & \to & \mathbb{A}^1/\mathbb{G}_m \\ \downarrow & & \downarrow \\ * & \to & B\mathbb{G}_m \end{array}$$
and comparison of $k[t^{\pm 1}]$-comodule structures)

·) 0^* given by $\otimes_{k[t]} k$ $\rightsquigarrow$ left square commutes

·) $(K_n)_{n \in \mathbb{Z}} \in \mathrm{Mod}_{k[t]}(D(k)^{\mathbb{Z}\text{-gr}})$

fib $(K_n \longrightarrow K_n[\frac{1}{t}])$ is t^∞ - torsion

$\rightsquigarrow \mathrm{colim}(K_n \xrightarrow{\cdot t} K_{n-1} \to \dots) = \mathrm{colim}(K_n[\frac{1}{t}] \xrightarrow{\cdot t} K_{n-1}[\frac{1}{t}] \to \dots)$

$\rightsquigarrow$ right square commutes

Rem $\exists$ version $/ \operatorname{Spec}(\mathbb{S})$ (Moulinos)

Def graded stack = stack $/ B\mathbb{G}_m$

filtered stack = stack $/ [\mathbb{A}^1/\mathbb{G}_m]$

Def Let $X \to [\mathbb{A}^1/\mathbb{G}_m]$ filtered stack. Then get diagram

ass. graded stack $\rightarrow$ underlying stack

$$\begin{tikzcd} X^{gr} \arrow[r] \arrow[d, "\pi^{gr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\pi"] & X^{u} \arrow[l] \arrow[d, "\pi^{u}"] \arrow[dl, phantom, "\llcorner", very near start] \\ B\mathbb{G}_m \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]} & * \arrow[l] \end{tikzcd} \qquad (*)$$

Def $X \xrightarrow{\pi} [\mathbb{A}^1/\mathbb{G}_m] \rightsquigarrow \mathcal{O}^{fil}(X) := \pi_* \mathcal{O}_X \in DF(k)$

If have base change for squares in $(*)$

associated graded of $\mathcal{O}^{fil}(X) = (\pi^{gr})_* \mathcal{O}_{X^{gr}}$

underlying objet $= \pi^u_* \mathcal{O}_{X^u}$

Base change ensured by $\quad$ X qcqs

(A) π of fin. coh. dim

$\forall R \in \mathrm{CAlg}_k^{an}$, $\operatorname{Spec} R \to [\mathbb{A}^1/\mathbb{G}_m]$

$\pi_{R,*} \; \mathrm{QCoh}(X \times_{\mathbb{A}^1/\mathbb{G}_m} \operatorname{Spec} R)_{\geq 0} \longrightarrow D(R)_{\geq n}$ for some n

(B) X admits flat hypercover by affines w/ flat transition maps

atlas

$\rightsquigarrow$ base change for right $\square$ $\mathrm{QCoh}(X)_{<\infty}$

left $\quad$ $\mathrm{QCoh}(X)$

__Classical situation:__ X normal variety, $X \longrightarrow [\mathbb{A}^1/\mathbb{G}_m]$ flat

$\rightsquigarrow L \in \mathrm{Pic}(X) + s \in H^0(X, L)$

$n \geq 1$

$\leftrightarrow D \subset X$ Cartier div

Filtration $\mathcal{O}^{fil}(X)$

$= \underline{\mathrm{Spec}}_X(\bigoplus L^{-n})$

$$\begin{tikzcd} V(L)\setminus 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & X \arrow[d] \\ \mathbb{A}^1 \arrow[r, "\rho"'] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd}$$

$\rightsquigarrow \rho^{-1}\mathcal{O}^{fil}(X) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, L^{-n})$

(derived)

$$\begin{tikzcd} D \arrow[r] \arrow[d] & X \arrow[d] & X \setminus D \arrow[l] \\ B\mathbb{G}_m \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]} & * \arrow[l] \end{tikzcd}$$

associated graded $= \bigoplus_{n \in \mathbb{Z}} \Gamma(D, L|_D^{-n})$

underlying object $= \Gamma(X \setminus D, L|_{X \setminus D})$

$\xrightarrow{s^{-1}} \Gamma(X \setminus D, \mathcal{O})$

$\rightsquigarrow$ filtration on $\Gamma(X \setminus D, \mathcal{O}_{X \setminus D})$

can be described concretely: $D \subset X$ Cartier div $\rightsquigarrow$ $\mathcal{O}_X(*D) \xrightarrow{\sim} \mathcal{O}_{X \setminus D}$ has pole-order filtr.

$\mathcal{O}_X(*D) = \bigcup \mathcal{O}_X(nD)$

Ex. ("split filtered stack") X stack, $\mathbb{G}_m \circlearrowright X$

$\rightsquigarrow$ filtered stack $X^{Fil} := [X \times \mathbb{A}^1 \underset{\Delta}{/} \mathbb{G}_m]$

$$\begin{tikzcd}
{[X/\mathbb{G}_m] = (X^{Fil})^{gr}} \arrow[r] & X^{Fil} \arrow[d] & (X^{Fil})^{u} = [X \times \mathbb{G}_m \overset{\text{diagonal}}{/} \mathbb{G}_m] \arrow[l] \arrow[d] \\
B\mathbb{G}_m \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]} & * \arrow[l]
\end{tikzcd}$$

$$= [X \times \mathbb{G}_m \underset{2^{nd}}{/} \mathbb{G}_m] = X$$

$\exists$ $\mathbb{G}_m$-equivar

$$\begin{array}{ccc}
\overset{\mathbb{G}_m \text{ diag}}{\circlearrowright} & & \overset{\mathbb{G}_m \text{ } 2^{nd}}{\circlearrowright} \\
X \times \mathbb{G}_m & \longrightarrow & X \times \mathbb{G}_m \\
(x, t) & \longmapsto & (xt^{-1}, t)
\end{array}$$

III Applications to Witt vectors

Fix prime p, $k = \mathbb{Z}_{(p)}$ Witt vectors = p-typical Witt vectors

Frob: $W \longrightarrow W \rightsquigarrow$ subgroup schemes Ker, Fix

__Goal__: Construct filtration on Fix w/ associated graded $[\mathrm{Ker}/\mathbb{G}_m]$

§1 Review Witt vectors

__Def__ (Joyal) A δ-ring is a pair (R, δ) where

- R comm. ring
- $\delta : R \to R$ map subject to some relations ensuring

Frob: $x \longmapsto x^p + p \cdot \delta(x)$ is a ring homom.

__Fact__: $\exists$ adjoint functors

$$\begin{tikzcd} (\delta\text{-Rings}) \arrow[r, "\mathrm{Forget}"] & (\mathrm{Rings}) \arrow[l, bend right, "\mathrm{Free}"'] \arrow[l, bend left, "W"] \end{tikzcd}$$

1) $\mathrm{Free}(k[x]) = k\{x\} = k[x_0, x_1, x_2, \ldots]$

$\delta(x_i) = x_{i+1}$

2) As sets,

$$\begin{tikzcd}
W(R) = \mathrm{Hom}_{\mathrm{Ring}}(k[x], W(R)) = \mathrm{Hom}_{\delta\text{-ring}}(k\{x\}, W(R)) \arrow[d] & \\
& = \mathrm{Hom}_{\mathrm{Ring}}(k\{x\}, R) = R^{\mathbb{N}} \arrow[d] \\
W_n(R) \arrow[r] & \mathrm{Hom}_{\mathrm{Ring}}(k[x_0, \ldots, x_n], R) = R^{n+1}
\end{tikzcd}$$

3) Have ring morphism

$$\mathrm{Ghost}: W(R) \longrightarrow R^{\mathbb{N}}$$

$$x \longmapsto (x, \underset{\substack{\| \\ x^p + p\delta(x)}}{\phi(x)}, \phi^2(x), \ldots)$$

<u>Thm</u> (Witt) Existence of Ghost uniquely determines on $W(R)$ (functorially in R)

4) $/\mathbb{Q}$: $\delta(x) = \dfrac{\phi(x) - x^p}{p}$

$\rightsquigarrow$ $\forall$ $\mathbb{Q}$-alg R, Ghost :
$$\begin{array}{ccc} W(R) & \xrightarrow{\sim} & R^{\mathbb{N}} \\ \downarrow & & \downarrow \\ W_n(R) & \longrightarrow & R^{n+1} \end{array}$$

5) $\exists$ multiplicative Teichmüller map $R \longrightarrow W(R) \overset{\text{sets}}{\simeq} R^{\mathbb{N}}$

$x \longmapsto (x, 0, 0, \dots) = [x]$

Everything functorial in R

$\rightsquigarrow$ comm. group scheme $W \simeq \lim W_n \overset{\text{as schemes}}{\simeq} \mathbb{A}^{n+1}$

+ Frobenius $\mathrm{Frob} :$
$$\begin{array}{ccc} W & \longrightarrow & W \\ \downarrow & & \downarrow \\ W_{n+1} & \longrightarrow & W_n \end{array}$$

+ Teichmüller map $[\,] : \mathbb{G}_m \longrightarrow W$

+ Ghost maps $W \longrightarrow \prod \mathbb{G}_a$

§2 Ker and Fix

<u>Def</u> $\mathrm{Fix} = \ker\left(W \xrightarrow{\mathrm{Frob} - \mathrm{id}} W \right)$

$\mathrm{Ker} = \ker\left(W \xrightarrow{\mathrm{Frob}} W \right)$

$/\mathbb{Q}$: Ghost:

$$\begin{tikzcd} W_{\mathbb{Q}} \arrow[r, "\sim"] \arrow[d, "\mathrm{Frob}-\mathrm{id}"'] & \prod \mathbb{G}_{a,\mathbb{Q}} \arrow[d] & (z_0, z_1, z_2, \dots) \arrow[d, mapsto] \\ W_{\mathbb{Q}} \arrow[r, "\sim"] & \prod \mathbb{G}_{a,\mathbb{Q}} & (z_1 - z_0, z_2 - z_1, z_3 - z_2, \dots) \end{tikzcd}$$

$\rightsquigarrow \mathrm{Fix}_{\mathbb{Q}} \simeq \mathbb{G}_{a,\mathbb{Q}} \overset{\Delta}{\hookrightarrow} \prod \mathbb{G}_{a,\mathbb{Q}}$

Likewise, $\mathrm{Ker}_{\mathbb{Q}} \simeq \mathbb{G}_{a,\mathbb{Q}} \underset{\mathrm{incl}_0}{\hookrightarrow} \prod \mathbb{G}_{a,\mathbb{Q}}$

<u>Goal</u>: Interpolate b/w Fix and Ker away from $\mathbb{Q}$

<u>Def</u>.

$$\begin{tikzcd}[column sep=small]
(x,\lambda) \arrow[rr, mapsto] & & ((\mathrm{Frob} - [\lambda^{p-1}])(x), \lambda) \\
G : W \times \mathbb{A}^1 \arrow[rr] \arrow[dr] & & W \times \mathbb{A}^1 \arrow[dl] \\
& \mathbb{A}^1 &
\end{tikzcd}$$

$\rightsquigarrow (\ker G) \subset W \times \mathbb{A}^1$

- $(\ker G)_0 = \mathrm{Ker}$

<u>Check</u>: $\ker G$ is $\mathbb{G}_m$-stable for diagonal action (via $[\,]$ on first comp.)

$\therefore$ For $\lambda \in \mathbb{A}^\times$, $(\ker G)_\lambda \simeq (\ker G)_1 \simeq \ker(\mathrm{Frob} - [1]) \simeq \mathrm{Fix}$

$$(x,\lambda) \longmapsto \left(\left[\tfrac{1}{\lambda}\right] x, 1\right)$$

$$\begin{tikzcd}
G : W \times \mathbb{A}^1 \arrow[r] \arrow[loop above, "{([\,],\, \mathrm{nat'l})}"] & W \times \mathbb{A}^1 \arrow[loop above, "{([\,]^p,\, \mathrm{nat'l})}"]
\end{tikzcd}$$

$(\mu x, \mu\lambda) \longmapsto (\mathrm{Frob}(\mu x) - [(\lambda\mu)^{p-1}](\mu x), \mu\lambda)$

$= [\mu^p]\,\mathrm{Frob}(x) - [\mu^p][\lambda^{p-1}](x), \mu)$

Apply split filtered stack construction to obtain fiber product diagram

<u>Def</u>

$$\begin{tikzcd}
H^{gr} = [\ker / \mathbb{G}_m] \arrow[r] \arrow[d] & H = [\ker G / \mathbb{G}_m] \arrow[d] & H^u = \mathrm{Fix} \arrow[l] \arrow[d] \\
{[W / \mathbb{G}_m]} \arrow[r] \arrow[d] & W^{Fil} := [W \times \mathbb{A}^1 / \mathbb{G}_m] \arrow[d] & W \arrow[l] \arrow[d] \\
B\mathbb{G}_m \arrow[r, "0"] & {[\mathbb{A}^1 / \mathbb{G}_m]} & * \arrow[l, "1"']
\end{tikzcd}$$

<u>Lemma</u> $G : W \times \mathbb{A}^1 \longrightarrow W \times \mathbb{A}^1$ is an fpqc cover, in part., have SES of group schemes$/\mathbb{A}^1$, $0 \to \ker G \to W \times \mathbb{A}^1 \xrightarrow{G} W \times \mathbb{A}^1 \to 0$.

<u>Cor</u> $H \to [\mathbb{A}^1/\mathbb{G}_m]$ flat

<u>Pf</u>
$$\begin{tikzcd}
\ker G \arrow[r] \arrow[d] & W \times \mathbb{A}^1 \arrow[d, "G"] \\
\mathbb{A}^1 \arrow[r] & W \times \mathbb{A}^1
\end{tikzcd}
\Rightarrow \ker G \to \mathbb{A}^1 \text{ flat} \Rightarrow \overset{= H}{[\ker G / \mathbb{G}_m]} \to [\mathbb{A}^1/\mathbb{G}_m] \text{ flat} \quad \square$$

Idea of pf of Lemma $\forall n$ $\exists$ factorization
$$\begin{tikzcd} W \times \mathbb{A}^1 \arrow[r, "G"] \arrow[d] & W \times \mathbb{A}^1 \arrow[d] \\ W_{n+1} \times \mathbb{A}^1 \arrow[r] & W_n \times \mathbb{A}^1 \end{tikzcd}$$

Limit argument $\rightsquigarrow$ STS $W_{n+1} \times \mathbb{A}^1 \longrightarrow W_n \times \mathbb{A}^1$ is fpqc cover

Critère de platitude par fibres $\Longrightarrow$ STS $(W_{n+1} \times \mathbb{A}^1)_\lambda \longrightarrow (W_n \times \mathbb{A}^1)_\lambda$ fpqc cover

$\forall \lambda \in \mathbb{A}^1 = \mathbb{A}^1_{\mathbb{Z}_{(p)}}$

Compatibility w/ $\mathbb{G}_m$-action $\Rightarrow$ Enough to check $\lambda = (T), (p, T), (T-1), (p, T-1)$

i.e., $W_{n+1,\mathbb{Q}} \xrightarrow{\text{Frob}} W_{n,\mathbb{Q}}$ $W_{n+1,\mathbb{F}_p} \xrightarrow{\text{Frob}} W_{n,\mathbb{F}_p}$ ✓ Kunz

$W_{n+1,\mathbb{Q}} \xrightarrow{\text{Frob}-\text{id}} W_{n,\mathbb{Q}}$ $W_{n+1,\mathbb{F}_p} \xrightarrow{\text{Frob}-\text{id}} W_{n,\mathbb{F}_p}$ ✓ Artin-Schreier

$/\mathbb{Q}$: can identify w/ $\prod \mathbb{G}_{a,\mathbb{Q}} \longrightarrow \prod \mathbb{G}_{a,\mathbb{Q}}$

$(z_0, z_1, z_2, \ldots) \longmapsto (z_1, z_2, \ldots) \,/\, (z_1 - z_0, z_2 - z_1, \ldots)$ $\square$